10-1-1959

Some Examples of Global Instability of the Competitive Equilibrium

Herbert E. Scarf

COWLES FOUNDATION DISCUSSION PAPER NO. 79

Some Examples of Global Instability of the Competitive Equilibrium*

by

Herbert Scarf

October 13, 1959

* Research undertaken by Stanford University under Contract Nonr-225(28), NR 047-019 with the Office of Naval Research and completed at the Cowles Foundation for Research in Economics under Contract Nonr-358(01), NR 047-066 with the Office of Naval Research.

Some Examples of Global Instability of the Competitive Equilibrium

by

Herbert Scarf

In this paper we consider the problem of stability of the competitive equilibrium. The market demand functions are sums of individual demand functions obtained directly by utility maximization. The rate of change of the price of each commodity is assumed to be proportional to the excess market demand for that commodity. A number of examples are given for which the motion of the prices is globally unstable in the sense that starting from any set of prices other than equilibrium, the prices oscillate without tending towards equilibrium.

I. Introduction

The problem of stability of the competitive equilibrium is described in [3], and we shall content ourselves with a review.* Several individuals

* The reader may wish to consult the excellent bibliography given in [3] and also in [1].

with utility functions $U_i(x_1, \ldots, x_n)$ for the same commodities are engaged in the trade of these commodities. Each individual begins the trading with an initial endowment of goods, say $I_1^i, I_2^i, \ldots, I_n^i$, for the i^{th} individual. An initial vector of prices $p_1, \ldots, p_n$ is announced, and each consumer then determines his demand for all of the commodities by the usual procedure of maximizing his utility function subject to the constraint that his expenditure shall not exceed the value of his initial endowment of goods at the stated price vector.

For each commodity the sum of the individuals' demand functions minus the sum of the initial endowments of that commodity is called the market excess demand for the commodity, and in this paper will be denoted by $f_i(p_1, \ldots, p_n)$, where the subscript refers to the commodity in question. (These functions are homogeneous of degree zero and satisfy the Walras Law $\Sigma p_i f_i \equiv 0$.) Excess demand functions for individuals will always be denoted by x_i.

An equilibrium price vector is, of course, a vector of prices for which all of the excess demand functions vanish, and recent work in this area [2, 8] has shown that under suitable regularity conditions equilibrium prices will always exist. (Because of the homogeniety of the demand functions, any multiple of an equilibrium is again an equilibrium, and in this sense we should speak of an equilibrium ray. There may be several such rays.) The stability problem, on the other hand, is less concerned with the existance of equilibrium, and more with the question of what happens to the prices if initially they are different from equilibrium.

There is nothing in the model described so far which enables us to compute the motion of prices if we are not at an equilibrium point. For this we need some specific assumptions on the price adjustment process, that is the procedure by which prices may be expected to change if we are away from an equilibrium price. The intuitive notion that an excess of supply over demand should result in a decrease in price, and an excess of demand should result in an increase in price has been formalized mathematically in [7] and [3] by the statement that

$$\frac{dp_i}{dt} = H_i\left[f_i(p_1, \ldots, p_n)\right], \text{ where } H_i \text{ is a sufficiently}$$

regular sign-preserving function of its argument. (These equations are to hold if all $p_i \geq 0$.) In order to be concrete, in this paper we shall generally take the functions H to be identically 1 , so that for each commodity the rate of change of price is equal to the excess demand.

Now let us turn our attention to the problem of stability with this type of adjustment process. Early work in this area [5, 6] tends to emphasize what might be called "local" stability; the initial prices are assumed to be close to some equilibrium point, and an analysis is made of whether there is a tendency to converge to the equilibrium point, depart from the equilibrium point, or perhaps even a tendency to more complex types of behavior. The "local" analysis proceeds by means of the linear terms of the Taylor series expansion about the equilbrium point, thus converting the problem to a linear differential system with constant coefficients. It is possible in this type of analysis to obtain many examples of completely unstable equilibria (though it should be mentioned that there is an uncomfortable tendency for examples to be produced without any consideration of their origins as market demand functions derived by the summation of individual demand functions). The local analysis, however, is somewhat unsatisfactory in that it is quite possible for other equilibrium points to exist, and the system cannot be said to be unstable without examining whether the prices tend to another equilibrium point.

This consideration leads naturally to the problem of stability in the "global" sense [3] which is concerned with the solution of the differential equations based on the price adjustment mechanism, for an arbitrary initial set of prices. If the solution of the differential equations approaches <u>some</u> equilibrium point as the time becomes infinite, then we have global stability. Clearly it is quite possible for these to be several equilibrium points, none of which is completely stable from the local point of view (that is, attracts all neighboring points) and where the system, in its entirety is globally stable.

Most of the results on global stability of the pure trade model discussed in this paper may be found in [1]. Aside from certain very special cases(global stability is known for a single consumer, and also for the case of two goods.), the most important result (obtained by Arrow, Block, and Hurwicz), is that global stability will occur if all of the goods are gross substitutes; mathematically this means $\frac{\partial f_i}{\partial p_j} > 0$ if $i \neq j$. (There are also some related results described in the above paper.) Aside from this, very little else has been found: no assumptions substantially different from gross substitutability have been shown to imply stability, and up to the present no examples of instability have been produced.

This paper presents a series of examples, all derived from utility maximization, which are globally unstable. The examples given here involve three consumers and three goods, but the techniques may be extended to either more consumers or more goods. The examples all involve a rather simple relationship between the utility functions of the three consumers, namely

$$(1) \qquad \begin{aligned} U_2(x_1, x_2, x_3) &= U_1(x_2, x_3, x_1) \qquad \text{and} \\ U_3(x_1, x_2, x_3) &= U_2(x_2, x_3, x_1) \quad , \end{aligned}$$

and corresponding cyclic permutations of the initial endowments. This relationship is introduced to simplify the calculations and as the method will show it may be relaxed.

It is an easy consequence of utility maximization that for the price adjustment process we have selected (rate of change of price equal to excess demand) the motion of the prices will always be constrained to the sphere $p_1^2 + p_2^2 + p_3^2 = \text{const}$. In all of our examples there will be an equilibrium point at $(p_1 = p_2 = p_3)$, which except for the first example will be locally completely unstable (there is a small region around this equilibrium point such that if the initial price is in the region, the prices will eventually leave this region and stay out.). In the first example the motion of the prices will be in concentric curves about the equilibrium point, whereas the more complex examples will give rise to limit cycles.

As we shall see from the examples, instability does not depend on a delicate assignment of values of initial stocks or parameters in the utility functions. It is a relatively common phenomenon, which seems to be related to divergent types of complementarity among the various consumers.

What are the implications of these examples? It seems to me that there are several possible interpretations that <u>might</u> be made.

1.- One possible interpretation is that the model is substantially realistic and that instabilities of the type described in this paper could possibly occur. An even more presumptuous interpretation along these lines is that instability is responsible for some aspects of the business cycle, though for this sort of interpretation it would seem to be necessary to produce examples of instability with a model of a complete economy rather than a pure trade model alone.

2.- Another possible interpretation is that the price adjustment process postulated above is not correct. This view can, of course, be held without any reference to the question of stability. An argument for this position is that in one sense or another we are considering a dynamic process and yet nowhere do the simplest dynamic considerations such as saving, interest, etc. appear in the model.

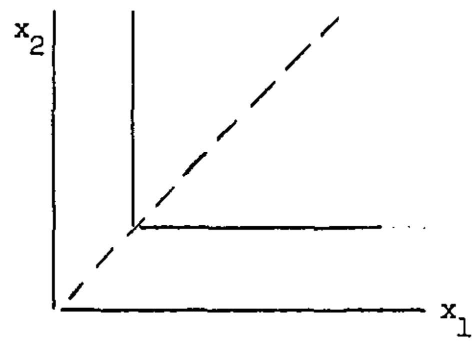
3.- As a final interpretation it might be argued that the types and diversities of complementarities exhibited in this paper do not appear in reality, and that only relatively similar utility functions should be postulated. This view may be substantiated by the known fact that if all of the individuals are identical in both their utility functions and initial holdings, then global stability obtains.

I would like to thank K. J. Arrow, L. Hurwicz, and H. Uzawa for a number of stimulating conversations.

II. A Very Simple Example of Instability

In this section we shall describe a very simple example which leads to instability. It will be seen that this example is quite an extreme one. In the next section we shall describe a number of additional examples which while somewhat more complex, do not have the disagreeable features of the present one.

Let the utility function of the first consumer be $U_1(x_1, x_2, x_3) = \min(x_1, x_2)$ and his initial endowments $(1, 0, 0)$. This consumer has no desire for the third good and his indifference curves for the first two goods are of the form:



For any income M the same quantity will be demanded of goods one and two and therefore the demand functions are

$$y_1(p_1, p_2, p_3, M) = \frac{M}{p_1 + p_2}$$

$$y_2(p_1, p_2, p_3, M) = \frac{M}{p_1 + p_2}$$

$$y_3(p_1, p_2, p_3, M) = 0 \quad .$$

Now the income of the first consumer is derived from his endowment of a single unit of good one so that $M = p_1$ and therefore the excess demand functions of the first consumer (supply is being subtracted off now) are given by

$$x_1 = \frac{-p_2}{p_1 + p_2}$$

$$x_2 = \frac{p_1}{p_1 + p_2}$$

$$x_3 = 0 \quad .$$

The excess demand functions of the second consumer are obtained by a cyclic permutation of all subscripts, $1 \longrightarrow 2$, $2 \longrightarrow 3$, $3 \longrightarrow 1$, and once more for the third consumer. If we add these together, the market excess demand functions are given by

$$f_1 = \frac{-p_2}{p_1 + p_2} + \frac{p_3}{p_3 + p_1}$$

$$f_2 = \frac{-p_3}{p_2 + p_3} + \frac{p_1}{p_1 + p_2}$$

$$f_3 = \frac{-p_1}{p_3 + p_1} + \frac{p_2}{p_2 + p_3} \quad ,$$

and of course the price adjustment process leads to the differential equations

$$\text{(2)} \qquad \frac{dp_i}{dt} = f_i\ (p_1, p_2, p_3)\ .$$

It is a trivial matter to verify that $p_1 = p_2 = p_3$ is the only equilibrium point. The fact that Σp_i^2 = const. follows from $\Sigma p_i \frac{dp_i}{dt} = 0$, which is the Walras Law. In order to show that the solutions are unstable we shall demonstrate that $p_1p_2p_3$ = const. for any solution of (2). This would follow if we could show that

$$\text{(3)} \qquad f_1p_2p_3 + f_2p_1p_3 + f_3p_1p_2$$

equals zero, but (3) is equal to

$$\frac{p_3(p_1^2 - p_2^2)}{p_1 + p_2} + p_2 \frac{(p_3^2 - p_1^2)}{p_3 + p_1} + p_1 \frac{(p_2^2 - p_3^2)}{p_2 + p_3} = p_3(p_1 - p_2) + p_2(p_3 - p_1)$$

$$+ p_1(p_2 - p_3) = 0\ .$$

That this implies instability is clear. Let the initial prices be chosen such that $p_1^2 + p_2^2 + p_3^2 = 3$ so that the intersection of the equilibrium ray and this sphere is (1, 1, 1). The value of $p_1p_2p_3$ at equilibrium is one, and therefore if the initial price gives a value different from one to $p_1p_2p_3$ we never reach equilibrium. It should be remarked that the maximum of $p_1p_2p_3$ subject to the constraint $\Sigma p_i^2 = 3$ is actually one, so that if the initial position is anything other than (1, 1, 1), the path is completely unstable.

Occasionally the following price adjustment process is discussed. One of the goods is singled out and kept constant; the remaining goods are meant to vary according to the differential equations given above. If in our case we put $p_3 \equiv 1$, then we are led to the system

$$\frac{dp_1}{dt} = \frac{-p_2}{p_1 + p_2} + \frac{1}{1 + p_1}$$

$$\frac{dp_2}{dt} = \frac{-1}{p_2 + 1} + \frac{p_1}{p_1 + p_2} \quad .$$

Routine calculations show that for this system

$$p_1p_2 e^{-\frac{1}{2}(p_1^2 + p_2^2)} = \text{const.},$$

and this is again sufficient to show instability.

III. A Class of Examples

The example of the previous section has a number of special properties. All of the Slutsky terms ($\frac{\partial y_i}{\partial p_j} + y_j \frac{\partial y_i}{\partial M}$) are zero, the indifferent surfaces aren't strictly convex, and certainly not differentiable, and finally the initial holdings are of a rather extreme type. It might be thought that one or several of these properties is responsible for the instability of the previous example, and that stability would return if these properties were removed. The examples of the present section show, however, that it is quite easy to obtain instability with none of the objectionable properties mentioned above.

We shall make an attempt to keep the reasoning relatively general in this section. Some conditions will be described which imply instability, and we shall demonstrate by specific examples that these conditions may be satisfied.

As was mentioned in the introduction [see equation (1)], the utility functions of the three consumers will be obtained by a cyclic permutation of the goods, and the initial endowments. This means that if the excess demand functions of the first consumer are represented by

$$x_1(p_1, p_2, p_3),\ x_2(p_1, p_2, p_3),\ x_3(p_1, p_2, p_3), \tag{4}$$

then the excess demand functions of the second consumer for the first, second, and third goods respectively will be given by

$$x_3(p_2, p_3, p_1),\ x_1(p_2, p_3, p_1),\ x_2(p_2, p_3, p_1) \tag{5}$$

and those of the third consumer by

$$x_2(p_3, p_1, p_2)\ ,\ x_3(p_3, p_1, p_2)\ ,\ x_1(p_3, p_1, p_2)\ . \tag{6}$$

(Continuity and differentiability properties of these functions will be assumed whenever necessary.)

The market excess demand functions are obtained by the summation of individual demands and are therefore given by:

$$f_1(p_1, p_2, p_3) = x_1(p_1, p_2, p_3) + x_3(p_2, p_3, p_1) + x_2(p_3, p_1, p_2)$$

$$(7) \quad f_2(p_1, p_2, p_3) = x_2(p_1, p_2, p_3) + x_1(p_2, p_3, p_1) + x_3(p_3, p_1, p_2)$$

$$f_3(p_1, p_2, p_3) = x_3(p_1, p_2, p_3) + x_2(p_2, p_3, p_1) + x_1(p_3, p_1, p_2) .$$

The differential equations of stability are, of course, given by $\frac{dp_i}{dt} = f_i(p_1, p_2, p_3)$, when all prices are non-negative.

As we mentioned previously the Walras Law($\Sigma p_i x_i = 0$ and its consequence $\Sigma p_i f_i = 0$) implies that $\frac{d}{dt}(\Sigma p_i^2) = 0$, so that the motion of the prices will always be on that sphere with center at the origin passing through the initial price vector. Let us assume that the initial prices have been selected such that $\Sigma p_i^2 = 3$. When we speak of equilibrium prices we shall mean the intersection of the equilibrium ray and this sphere.

Lemma 1. The price vector (1, 1, 1) is an equilibrium price.

This is immediately obvious by an application of the Walras Law to equations (7), when all of the prices are set equal to one.

It is not at all correct, for a general selection of the individual demand functions x_1, x_2, x_3, that the market demand functions f_1, f_2, f_3 (7), must have a unique equilibrium point at (1, 1, 1). There are a number of very simple conditions, however, which imply that this equilibrium is unique. We shall give only one such set of conditions. The reader may easily think of others.

Lemma 2. Let

$$A = \frac{\partial x_1}{\partial p_2} + \frac{\partial x_2}{\partial p_3} + \frac{\partial x_3}{\partial p_1} < 0 \qquad \text{and}$$

$$B = \frac{\partial x_1}{\partial p_3} + \frac{\partial x_2}{\partial p_1} + \frac{\partial x_3}{\partial p_2} > 0 \quad , \quad \text{everywhere in the orthant}$$

$p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$.

Then the equilibrium point (1, 1, 1) is unique (aside from constant multiples).

In order to demonstrate this lemma, let us first make the observation that

$$f_2(p_1, p_2, p_3) = f_1(p_2, p_3, p_1) ,$$

(8)
$$f_3(p_1, p_2, p_3) = f_2(p_2, p_3, p_1) , \quad \text{and}$$

$$f_1(p_1, p_2, p_3) = f_3(p_2, p_3, p_1) .$$

This implies that if (α, β, γ) is an equilibrium point, then so is (β, γ, α) , and also (γ, α, β). Therefore if we have an equilibrium point different from (1, 1, 1), then we may find an equilibrium point (α, β, γ) , perhaps by permutation, with either

(9)
$$\alpha \geq \beta \geq \gamma$$

(at least one of the inqualities is strict.), or else

(10)
$$\alpha \leq \beta \leq \gamma$$

(again the same remark.).

Let us assume that the former holds. An argument similar to the one we are about to give, works if instead of (9) we have (10).

We shall show that $f_2(\alpha, \beta, \gamma)$ must be different from zero. Now

(11)
$$f_2(\alpha, \beta, \gamma) = x_2(\alpha, \beta, \gamma) + x_1(\beta, \gamma, \alpha) + x_3(\gamma, \alpha, \beta) .$$

The assumptions of this lemma imply that as a function of α, f_2 is strictly increasing, and as a function of γ, f_2 is strictly decreasing. Since $\alpha \geq \beta \geq \gamma$, (with inequality somewhere) we have

$$(12) \qquad f_2(\alpha, \beta, \gamma) > x_2(\beta, \beta, \beta) + x_1(\beta, \beta, \beta) + x_3(\beta, \beta, \beta) = 0 ,$$

by the Walras Law. This completes the proof of the lemma. (It might be remarked that other conditions for uniqueness can be given in terms of linear combinations of the f's .)

Subsequently, in this paper, we shall have occasion to consider specific examples of demand functions for which the conditions of lemma 2 do not hold for all prices in the orthant $p_1 \geq 0$, $p_2 \geq 0$, $p_3 \geq 0$, but rather for certain subsets. We shall then have recourse to the following lemma which involves a type of subset convenient for us. The lemma is demonstrated in the same manner as lemma 2. It should be noted that the operations involved in the proof of lemma 2 do not take us out of the set described in lemma 3.

Lemma 3. Let the conditions of lemma 2, hold for the set of prices $\left(\frac{p_1}{p_3} > \epsilon, \ \frac{p_2}{p_1} > \epsilon, \ \frac{p_3}{p_2} > \epsilon\right)$, where ϵ is a small positive number. Then there are no equilibrium points in this set other than (1, 1, 1) and its multiples.

We shall now give a condition on the individual demand functions x_1, x_2, x_3 which implies that the equilibrium point (1, 1, 1) is unstable.

Lemma 4. If

$$C = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_2}{\partial p_2} + \frac{\partial x_3}{\partial p_3}$$

is positive, at the point (1, 1, 1) , then this equilibrium point is locally unstable in the following sense: There is a region (on the sphere $\Sigma p_i^2 = 3$) about the point (1, 1, 1) such that any point in this region (other than the equilibrium point) will move away from the point (1, 1, 1).

The Jacobian of the functions f_1, f_2, f_3, at the point (1, 1, 1) is given by

$$\begin{pmatrix} C & A & B \\ B & C & A \\ A & B & C \end{pmatrix} , \tag{13}$$

all evaluated at that particular point. Moreover, the functions x_1, x_2, x_3 are homogeneous of degree zero, and therefore the Euler relationship, at the point (1, 1, 1) tells us that

$$A + B + C = 0 . \tag{14}$$

The Jacobian, therefore, has one characteristic root equal to zero. The other characteristic roots may be shown to have positive real parts, because of the assumption $C > 0$. It is then possible to apply standard theorems of differential equations [see 4, chap. 13], in order to deduce local instability. However the various properties of demand functions permit a simple independent proof of this fact, which we shall reproduce here.

Let

$$V(p_1 p_2 p_3) = \frac{1}{2} \Sigma (p_i - 1)^2 , \text{ so that} \tag{15}$$

$$\frac{dV}{dt} = \Sigma (p_i - 1) \frac{dp_i}{dt} = - \Sigma f_i . \tag{16}$$

We shall show that $\Sigma f_i < 0$ in a small region on the sphere $\Sigma p_i^2 = 3$ about (1, 1, 1), (except for this point itself) and this will demonstrate the lemma. Of course at (1, 1, 1), $\Sigma f_i = 0$. Using the Taylor series expansion, we obtain

$$\Sigma f_i = \sum_j (p_j - 1) \sum_i \frac{\partial f_i}{\partial p_j} + \frac{1}{2} \sum_j \sum_k (p_j - 1)(p_k - 1) \sum_i \frac{\partial^2 f_i}{\partial p_j \partial p_k} + o(V^{3/2}) , \tag{17}$$

where all of the partial derivatives are evaluated at the point (1, 1, 1). Let us simplify some of the terms. The Walras Law reads $\Sigma p_i f_i \equiv 0$, so that differentiating with respect to p_j, we obtain

$$f_j + \sum_i p_i \frac{\partial f_i}{\partial p_j} \equiv 0 . \tag{18}$$

At (1, 1, 1), this implies $\sum_i \frac{\partial f_i}{\partial p_j} = 0$, and therefore the linear terms of the Taylor series vanish. Differentiating (18) once more with respect to p_k, we obtain, at the point (1, 1, 1),

$$\sum_i \frac{\partial^2 f_i}{\partial p_j \partial p_k} = -\left(\frac{\partial f_j}{\partial p_k} + \frac{\partial f_k}{\partial p_j}\right). \tag{19}$$

From (14), (17), and (19) we see that

$$\sum_i f_i = -\frac{1}{2} \sum_{j,k} (p_j - 1)(p_k - 1) c_{jk} + o(V^{3/2}) , \tag{20}$$

where

(21) $$(c_{jk}) = \begin{pmatrix} 2C, & -C, & -C \\ -C, & 2C, & -C \\ -C, & -C, & 2C \end{pmatrix} \quad .$$

Therefore

(22) $$\Sigma f_i = -\frac{C}{2}\left\{(p_1 - p_2)^2 + (p_2-p_3)^2 + (p_3-p_1)^2\right\} + o(V^{3/2}) \quad .$$

But I claim that on the surface $\Sigma p_i^{\ 2} = 3$, with all $p_i \geq 0$, we always have

(23) $$(p_1 - p_2)^2 + (p_2 - p_3)^2 + (p_3 - p_1)^2 \geq 3/2\ \Sigma\ (p_i - 1)^2$$

Accepting this as correct for the moment, we see that

(24) $$\Sigma f_i < -\frac{3}{2}\, C\, V + o(V^{3/2}) \quad ,$$

and therefore Σf_i is negative for V sufficiently small.

Equation (23) is demonstrated as follows:

First of all on this sphere

(25) $$3 \geq \Sigma p_i \quad .$$

Multiplying by Σp_i and expanding, we obtain

(26) $$3\ \Sigma p_i \geq p_1^{\ 2} + p_2^{\ 2} + p_3^{\ 2} + 2(p_1p_3 + p_1p_2 + p_2p_3)$$

and by subtracting 9 from each side we obtain

(27) $$\frac{3}{2}\ (-\Sigma p_i^{\ 2} + 2\Sigma p_i - 3) \ \geq\ -2\ (p_1^{\ 2} + p_2^{\ 2} + p_3^{\ 2}) + 2\ (p_1p_3 + p_1p_2 + p_2p_3)$$

which is the same as (23). This finishes the proof of Lemma 4.

We shall shortly exhibit some excess demand functions, derivable from a utility function, which satisfy the hypothesis of both Lemmas two and four. They will provide us with examples of a unique equilibrium point, which is unstable locally. Does this imply global instability? In a certain sense this becomes a matter of definition, the problem being that it is possible that one of the prices might become zero. In this event, some care should be taken since the differential equations (2) no longer describe the motion of the prices. In our examples, we shall use special techniques to demonstrate that the paths actually stay away from the boundary of the positive orthant.

Now let us consider utility functions for the first consumer of the form

$$U(x_1, x_2, x_3) = -\left(\frac{\alpha_1^{a+1}}{x_1^{a}} + \frac{\alpha_2^{a+1}}{x_2^{a}} + \frac{\alpha_3^{a+1}}{x_3^{a}}\right) , \tag{28}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is an arbitrary non-negative vector, and a is a positive constant. Let the initial endowment of the first consumer be represented by (I_1, I_2, I_3). As before we permute the utility function cyclically for the second and third consumers.

We shall show that instability arises, whenever $a > 1$, $(\alpha_1,\alpha_2,\alpha_3)$ is close to $(b, 1, 0)$ with $b > \frac{a+1}{a-1}$, and (I_1,I_2,I_3) is close to $(1, 0, 0)$. (The specific meaning of close will be clarified.) This will give us examples with none of the disagreeable features of the simple examples in section II. Let us first examine, in detail, the case

$$(\alpha_1, \alpha_2, \alpha_3) = (b, 1, 0) \qquad \text{and}$$

$$(I_1, I_2, I_3) = (1, 0, 0) \qquad , \tag{29}$$

with

$$b > \frac{a+1}{a-1} \quad .$$

A routine calculation shows us that the excess demand functions for the first individual are given by:

$$x_1 = \frac{bp_1^{\frac{a}{1+a}}}{bp_1^{\frac{a}{1+a}} + p_2^{\frac{a}{1+a}}} - 1$$

(30)
$$x_2 = \frac{1}{p_2^{\frac{1}{1+a}}} \quad \frac{p_1}{bp_1^{\frac{a}{1+a}} + p_2^{\frac{a}{1+a}}}$$

$$x_3 = 0$$

Let us show that the conditions of Lemma 2 are verified. We have

(31)
$$A = \frac{\partial x_1}{\partial p_2} + \frac{\partial x_2}{\partial p_3} + \frac{\partial x_3}{\partial p_1}$$

$$= \frac{\partial x_1}{\partial p_2} < 0 \qquad , \qquad \text{and}$$

(32)
$$B = \frac{\partial x_1}{\partial p_3} + \frac{\partial x_2}{\partial p_1} + \frac{\partial x_3}{\partial p_2}$$

$$= \frac{\partial x_2}{\partial p_1} > 0 \quad .$$

Therefore the system of market demand functions (7) has a unique equilibrium point given by (1, 1, 1). In order to show that it is unstable we compute

$$(33) \qquad C = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_2}{\partial p_2} + \frac{\partial x_3}{\partial p_3} \quad \text{at} \quad (1, 1, 1) \text{ , or}$$

$$(34) \qquad C = \frac{ab - (b+1) - a}{(b+1)^2 \, (a+1)} \qquad ,$$

which is positive if $b > \frac{a+1}{a-1}$. It follows from Lemma 4 that for this relationship between a and b the unique equilibrium point is also locally unstable.

The next step will be to show that for this example the prices stay away from the boundary of the positive orthant. This will be a consequence of the following lemma.

Lemma 5. Let

$$\varphi = \operatorname{Min}\left(\frac{p_2}{p_1}, \frac{p_3}{p_2}, \frac{p_1}{p_3}\right).$$ There is a small positive constant K , such that if

$$\varphi < K \text{ , then } \frac{d\varphi}{dt} > 0 \text{ .}$$

Let us assume that the minimizing value above is $\frac{p_1}{p_3} = c$, say. The other cases are handled by a cyclic permutation. We want to show that $\frac{d}{dt}\left(\frac{p_1}{p_3}\right) > 0$ or $p_3 f_1 - p_1 f_3 > 0$, if c is sufficiently small. If f_1 and f_3 are computed according to (7) , we see that $p_3 f_1 - p_1 f_3$ has the sign of

$$(35) \qquad \frac{\left(\frac{p_3}{p_1}\right)^2 + 1}{b\left(\frac{p_3}{p_1}\right)^{\frac{a}{1+a}} + 1} - \frac{\frac{p_3}{p_1}}{b\left(\frac{p_1}{p_2}\right)^{\frac{a}{1+a}} + 1} - \frac{\frac{p_2}{p_3}}{b\left(\frac{p_2}{p_3}\right)^{\frac{a}{1+a}} + 1}$$

The first term of (35) is equal to

$$\frac{\left(\frac{1}{c}\right)^2 + 1}{b\left(\frac{1}{c}\right)^2 + 1} \qquad ,$$

and since $\frac{p_2}{p_3} < \frac{1}{c}$, we have

$$\frac{-\frac{p_2}{p_3}}{b\left(\frac{p_2}{p_3}\right)^{\frac{a}{1+a}} + 1} > \frac{\frac{1}{c}}{b\left(\frac{1}{c}\right)^{\frac{a}{1+a}} + 1} \qquad .$$

Also we have

$$\frac{-\frac{p_3}{p_1}}{b\left(\frac{p_1}{p_2}\right)^{\frac{a}{1+a}} + 1} > -\frac{1}{c} \qquad .$$

Adding these terms together we see that (35) will be positive if

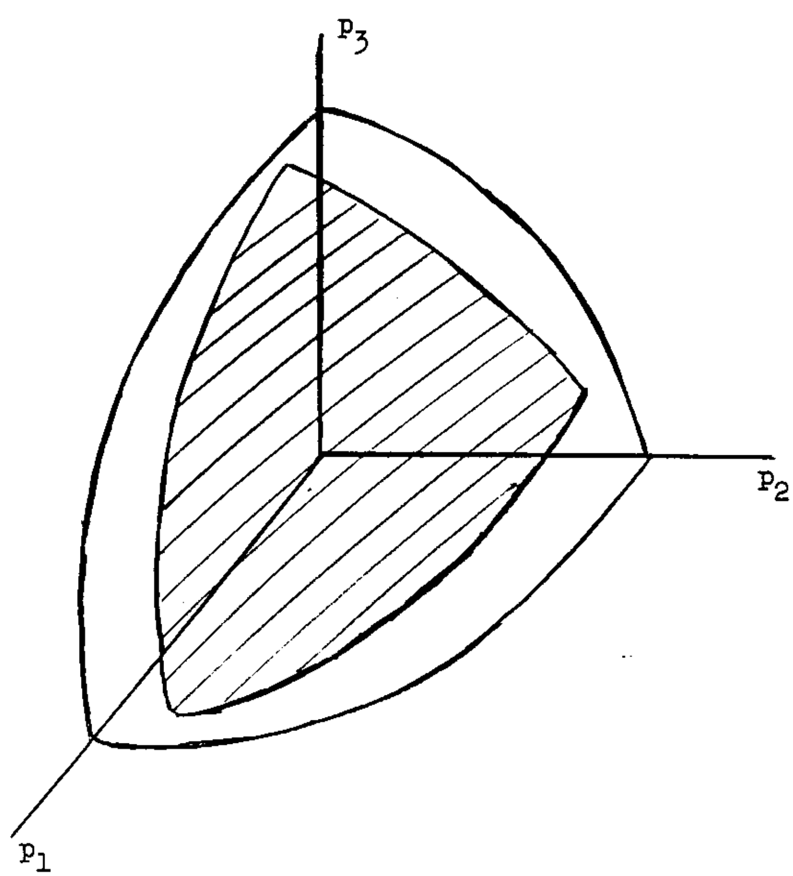
$$\left(\frac{1}{c}\right)^2 + 1 \quad \frac{2}{c} - b\left(\frac{1}{c}\right)^{1 + \frac{a}{1+a}} > 0 \quad ,$$

and this is certainly correct for small c . This demonstrates Lemma 5.

These observations, taken together, show us that the system of demand functions defined by (29) give rise to global instability of the price adjustment mechanism.

One more remark is in order. I claim that if we take an excess demand which is close to (29), on a large proper subset of the positive orthant, (and has derivatives which are close) and form the market demand functions using the above method of cyclic permutation, then this will also give rise to global instability. (Such an example may be obtained by taking $(\alpha_1, \alpha_2, \alpha_3)$ close to $(b, 1, 0)$ and (I_1, I_2, I_3) close to $(1, 0, 0)$, or in other ways.)

In order to see this let us interpret Lemma 5. geometrically.



The figure represents the surface of the sphere $\Sigma p_i^2 = 3$, and the shaded region, the set of points with

$$\text{Min} \quad \frac{p_2}{p_1} \; , \frac{p_3}{p_2} , \; \frac{p_1}{p_3} \; \geq \; K \; ,$$

for a small value of K . Lemma 5. tells us that a path which begins in this region, with K sufficiently small, will never leave the region. This is true because on the boundary of this region, expressions such as $p_3f_1 - p_1f_3$ are strictly positive. But this later fact will also be true for market demand functions based on individual demand functions close to (29).

It will also be true that inside of this region there will be no equilibrium points other than (1, 1, 1). For if the new demand functions and their derivatives are close to the corresponding quantities for (29) inside of the shaded region, then A for the new demand functions will be negative, and since it is homogeneous of degree -1 (being a derivative) A will be everywhere negative in the region defined by

$$\frac{p_2}{p_1} > K \, , \quad \frac{p_3}{p_2} > K \, , \quad \frac{p_1}{p_3} > K \; .$$

A similar statement with a reversed inequality holds for B , and therefore Lemma 3. may be applied to show that there are no equilibrium points other than (1, 1, 1) in the shaded region.

A similar argument convinces us of the local instability of the equilibrium point (1, 1, 1). It should be remarked that even though we have said nothing about possible equilibrium points outside of the shaded region for the new demand functions, a path beginning inside of the shaded region will not leave.

REFERENCES

[1] Arrow, Kenneth J., H. D. Block, and Leonid Hurwicz, "On the Stability of the Competitive Equilibrium, II," Econometrica, 27, 1959, pp. 82-109.

[2] ______________, and Gerard Debreu, "Existance of an Equilibrium for a Competitive Economy," Econometrica, 22, 1954, pp. 256-290.

[3] ______________, and Leonid Hurwicz, "On the Stability of the Competitive Equilibrium, I," Econometrica, October, 1958.

[4] Coddington, Earl A., and Norman Levinson, Theory of Ordinary Differential Equations, New York: McGraw-Hill, 1955.

[5] Hicks, John R., Value and Capital, Oxford: Oxford University Press, 1939.

[6] Metzler, Lloyd, "Stability of Multiple Markets: the Hicks Condition," Econometrica, 13, 1945, pp. 277-292.

[7] Samuelson, Paul A., "The Stability of Equilibrium: Comparative Statics and Dynamics," Econometrica, 9, 1941, pp. 97-120.

[8] Uzawa, Hirofumi, "Walras Tatonnement in the Theory of Exchange," Technical Report No. 69, Project on Efficiency in Decision Making in Economic Systems, Stanford University, March 1959.